

No-signaling from Gleason non-contextuality

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The no-signaling principle is shown to be a consequence of, but to not imply, Gleason non-contextuality, which may be considered as the more basic feature of quantum mechanics.

I. INTRODUCTION

Gleason's theorem [1] asserts that for a Hilbert space of dimension 3 or greater, the only possible probability measure $\mu(\alpha)$ associated with a particular linear subspace α of a Hilbert space will have the form $\text{Tr}(\hat{\Pi}_\alpha \rho)$, where $\hat{\Pi}_\alpha$ is the projector to the subspace, and ρ is the density matrix for the system. The premise needed for proving the theorem, apart from certain continuity requirements, is the assumption of non-contextuality: the probability measure, $\sum_j \mu(e_j)$, associated with a Hilbert subspace, is independent of the choice of basis (context), $\{e_j\}$. The no-signaling theorem as applicable to a composite system, essentially asserts that in a bipartite composite system, consisting of parts A and B , the reduced density operator ρ_A is unaffected by local operations in B [2].

For our purpose, it will be useful to restate these two principles as follows. Given a system S , we say that it is *tensor-product* partitioned into sectors J_i if the respective Hilbert spaces satisfy

$$\mathcal{H}_S = \bigotimes_i \mathcal{J}_i, \quad (1)$$

with $\dim(\mathcal{H}_S) = \prod_i \dim(\mathcal{J}_i)$. The no-signaling theorem asserts that the marginal probability distribution p_i associated with sector i is unaffected by local operations (in particular, measurement in some basis) in other sectors i' (i.e., operations of the form $I_{j \neq i'} \otimes O_{i'}$). It is customary to think of the sectors J_i as being spatially separated in order to make the term 'signaling' meaningful, though spatiality is not essential to the formalism. No-signaling applies at the single-particle level also, as discussed below.

We say that it is *tensor-sum* partitioned into sectors K_j if the respective Hilbert spaces satisfy

$$\mathcal{H}_S = \bigoplus_j \mathcal{K}_j \quad (2)$$

with $\dim(\mathcal{H}_S) = \sum_j \dim(\mathcal{K}_j)$. The Gleason non-contextuality assumption asserts that the probability measure q_j associated with sector K_j is unaffected by local operations in that sector, i.e., rotations of the basis vectors such that \mathcal{K}_j is an invariant subspace of the operations. In other words, the choice of measurement basis in that sector, or by extension, the projectors used to complete the full basis in the other sectors, does not alter the probability measure associated with K_j .

Expressed thus, the result we wish to prove, which is that no-signaling in a multi-partite system is a manifestation of single system non-contextuality in a tensor product setting, seems tantalizingly plausible. Stated differently, we wish to show that the mutual independence of marginal probabilities under local transformations in distinct sectors across a tensor *product* cut reduce to the independence of probability measures across a tensor *sum* cut.

II. SINGLE-PARTICLE CASE

Suppose we are given the orthogonal states of a qutrit, whose state space is spanned by the basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Alice and Bob are two spatially separated observers. Incoming qutrits prepared in the state $|\psi\rangle = \sum_j \beta_j |j\rangle$ ($j = 0, 1, 2$

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and $\sum_j |\beta_j|^2 = 1$) are first subjected to a *non-maximal test* corresponding to the measurement of the degenerate observable $X = a|0\rangle\langle 0| + b(|1\rangle\langle 1| + |2\rangle\langle 2|) = a|0\rangle\langle 0| + b(|+\rangle\langle +| + |-\rangle\langle -|)$, where $|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$ and a, b are real numbers. Alice is located in station A where she receives the qutrit if measurement of X returns a . Otherwise the particle goes to Bob, who measures it either in the basis $Y_1 \equiv \{|0\rangle, |1\rangle, |2\rangle\}$ or in the basis $Y_2 \equiv \{|0\rangle, |\pm\rangle\}$.

To show that non-contextuality entails no-signaling at the single-particle level, assume that there is a nonlocal signal from Bob to Alice, depending on whether he measures in the basis Y_1 or Y_2 . An instance of signaling implies that the probability $\text{prob}(|0\rangle)$ that Alice finds the particle depends on Bob's choice of basis in his subspace [3]:

$$\text{Prob}(|0\rangle|Y_1) \neq \text{Prob}(|0\rangle|Y_2).$$

But this means that the probability measure associated with the subspace $\text{span}\{|1\rangle, |2\rangle\}$ depends on the choice of basis, and therefore provides a mechanism by which the probability $\text{Prob}(|0\rangle)$ is contextual, and we obtain our required result.

Non-contextuality also implies the no-disturbance principle [4], which may be thought of as non-contextuality applied to whole observables rather than individual events or projectors. For, by construction,

$$[X, Y_j] = 0; [Y_1, Y_2] \neq 0. \quad (3)$$

Thus, if there were disturbance, namely the X marginals satisfy $\sum_i P(X = x, Y_1 = i) \neq \sum_j P(X = x, Y_2 = j)$, this would constitute context dependence.

III. MULTI-PARTICLE CASE

To prove that non-contextuality implies no-signaling in a tensor product setting, we consider a bi-partite system, given by the Hilbert space $\mathcal{H}_S \equiv \mathcal{H}_A \otimes \mathcal{H}_B$. Our result can straightforwardly be generalized to larger systems. By assumption of tensor product structure, a local operation on A 's side (a unitary operation followed by projective measurement or a POVM) has the form $\omega \equiv \mathcal{E}_A \otimes I_B$, where I_B is the identity operation in \mathcal{H}_B . And similarly ω' corresponding to another local operation on A 's side. Given any tensor-sum partition:

$$\mathcal{H}_S = \bigoplus_j \mathcal{K}_j \equiv \bigoplus_j \mathcal{H}_A \otimes \mathcal{B}_j, \quad (4)$$

where $\mathcal{H}_B = \bigoplus_j \mathcal{B}_j$, let δ_j denote the set of dimensions that span \mathcal{B}_j . Each partition \mathcal{K}_j is an invariant subspace under ω and under ω' . By non-contextuality, the probability measure μ associated with any given \mathcal{K}_j should be independent of the application of ω or ω' : i.e.,

$$\mu(\mathcal{K}_j|\mathcal{E}) = \mu(\mathcal{K}_j|\mathcal{E}') \equiv \mu(\mathcal{K}_j). \quad (5)$$

Now,

$$\mu(\mathcal{K}_j|\mathcal{E}) \equiv \sum_a \sum_{k \in \delta_j} \text{Prob}(A = a, B = k) = \text{Prob}_B(j|\mathcal{E}) \quad (6a)$$

$$\mu(\mathcal{K}_j|\mathcal{E}') \equiv \sum_{a'} \sum_{k \in \delta_j} \text{Prob}(A' = a', B = k) = \text{Prob}_B(j|\mathcal{E}'), \quad (6b)$$

where $\text{Prob}_B(j|\xi)$ is the probability for Bob to obtain outcome j in the ξ -context ($\xi = \mathcal{E}, \mathcal{E}'$). If signaling were possible, it would mean that there is a j such that

$$\text{Prob}_B(j|\mathcal{E}) \neq \text{Prob}_B(j|\mathcal{E}'), \quad (7)$$

which in view of Eq. (6), implies

$$\mu(\mathcal{K}_j|\mathcal{E}) \neq \mu(\mathcal{K}_j|\mathcal{E}'). \quad (8)$$

Together with Eq. (5), this implies a violation of non-contextuality in sector \mathcal{K}_j . This proves our stated result, which, as it happens, connects the Born rule to no-signaling via Gleason's theorem. An instance of contextuality, on the other hand, does not necessarily lead to signaling between spatially separated sectors or across a tensor product cut. For example, contextuality in a system with prime-numbered dimensionality cannot be represented as a signaling across

two tensor product sectors of non-trivial dimensionality. We may regard signaling as the avatar of contextuality in a spatial situation where some events correspond to geographically separated locations. Non-contextuality then is a stronger condition than no-signaling in this sense.

A proof of no-signaling also obtains as a special case of no-disturbance, where X and either Y_j in Eq. (3) are assumed to correspond to two different particles (tensor product sectors). However, when X and Y_j belong to the same particle, as we saw, X must be non-maximal or degenerate (at least in the subspace where Y_1 and Y_2 fail to commute). On the other hand, no such restriction appears when X and Y_j pertain to distinct particles or degrees of freedom or tensor-product sectors. This extra difference between the single-particle and multi-particle situation does not appear in the earlier proof, where the more elementary events rather than observables are the primary objects, and thus the reduction is unconditional.

IV. DISCUSSION AND CONCLUSIONS

We may regard non-contextuality as a more fundamental principle than no-signaling for several reasons. It pertains to a single system rather than a composite system. It enables unifying no-signaling into a single stronger no-go principle, as noted above. Moreover, no-signaling in our present sense arises in non-relativistic quantum mechanics. For it to be aligned with *relativistic* causality would be an odd conspiracy, that would need further explanation. No such difficulty arises when we regard non-contextuality as the more fundamental principle, with no-signaling an ‘innocent’ consequence of imposing it on a tensor-product structured space.

While no-signaling is no doubt a useful thumb-rule in deriving other results (e.g., as in Ref. [5, 6]), we believe that our observation would be of interest in axiomatic studies where no-signaling is treated as a primary postulate [7, 8]. It can also help clarify how, if potential violations of no-signaling exist in a more general theory than quantum mechanics, they may reduce or relate to single-particle effects. In Ref. [9], it is suggested that quantum optics is testably such a more general theory, with peculiarities introduced because of the lower-boundedness of energy imposed by the vacuum state.

Gleason non-contextuality is intimately related to, and yet quite distinct from, Kochen-Specker (KS) contextuality [10, 11], on which we report elsewhere [12]. To use existing terminology (as usually applied to quantum nonlocality in the multi-partite situation), Gleason non-contextuality refers to *parameter independence* [13] or signal locality, while KS contextuality to *outcome dependence* [13] or violation of Einstein locality. It is the latter that makes the former surprising. Parameter independence by itself would demand no more than ‘garden variety’ classicality, and outcome dependence by itself would demand superluminal classicality. It is putting them together that requires the subtle richness that is quantum contextuality or quantum nonlocality, or any other generalized non-signaling probability distributions [14, 15].

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